# Finding the Distance Function in the Poincaré Disk using Stereographic Projection

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**Abstract:** The Poincaré disk is often taught to students to provide a model for hyperbolic geometry. Despite its apparent simplicity, certain properties of the Poincaré disk still *appear* arbitrary. In particular, the Poincaré disk distance function $d(A, B) = \left|\ln\left(\frac{AQ}{BQ} \times \frac{BP}{AP}\right)\right|$ appears to be an overly-complicated and arbitrary choice. This paper sheds light on where the Poincaré disk distance function comes from to explain why it's not an arbitrary choice whatsoever. Specifically, I describe how to stereographically project a sphere onto a plane. Then, I show how to get the Poincaré disk using the same projection method, but starting with a hyperboloid in Minkowski space. Finally, I show that the Poincaré disk distance function is the projected distance function from the hyperboloid.

## 1. Introduction

The Poincaré disk is taught in geometry classes for good reason. As a model of hyperbolic geometry, the Poincaré disk is simple enough to draw on a piece of paper (or show on a computer screen [1]), yet weird enough to clearly show differences with Euclidean geometry. Students can visually show that lines curve inwards towards the center, Euclid's parallel postulate fails (more than one parallel line in Figure 1A), triangles appear squished down (angle sum $< 180°$ in Figure 1B), quadrilaterals appear squished down (angle sum $< 360°$ in Figure 1C), and more [2]. With the Poincaré disk, students are not told the properties of hyperbolic geometry – they discover them.

However, there is a non-Euclidean elephant in the room. The distance function in the Poincaré disk, stated as [2]

$$d(A, B) = \left|\ln\left(\frac{AQ}{BQ} \times \frac{BP}{AP}\right)\right| \tag{1}$$

(visually shown in Figure 2) elicits many furrowed eyebrows among students. They ask: everything was so simple, what went wrong? How could the distance between points $A$ and $B$ in

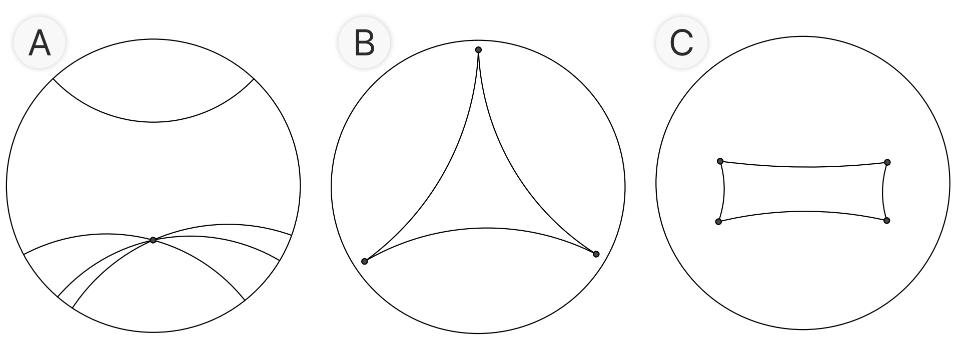


Figure 1: Simple constructions in the Poincaré disk to show deviations from Euclidean geometry.

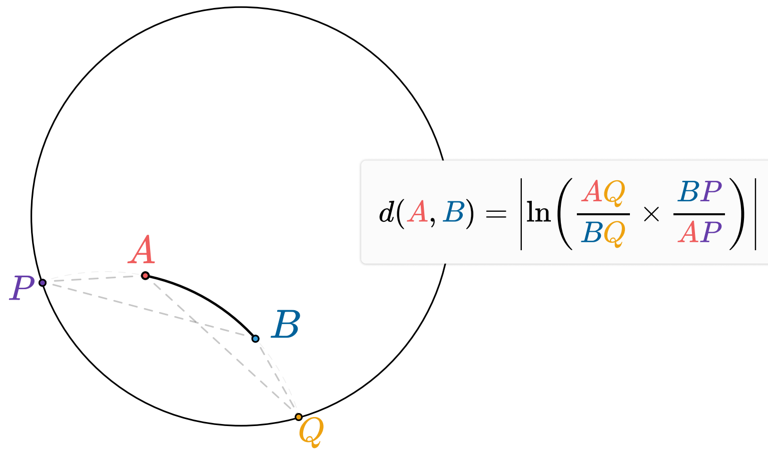


Figure 2: Example of the distance function in the Poincaré disk.

the Poincaré disk be defined by the ratio of the Euclidean distance between boundary points $P$ and $Q$ (see Figure 2), then evaluated inside of a natural log? The distance function *appears* to be overly complicated, completely arbitrary, and altogether confusing.

On the contrary! The distance function (1), starting from the right context, makes a great deal of sense! By showing where the distance function originally comes from, the reader can have a greater intuition for why the Poincaré distance function looks like it does. To be specific, this paper shows that the Poincaré disk distance function (1) comes from projecting a three-dimensional hyperboloid in Minkowski space onto the $xy$ plane.

This paper contributes the following explanations:

- **Why stereographic projection?** A scenario for the reader to imagine in Section 2 to show the advantages of stereographic projection.

- **Stereographic projection of a sphere** onto the plane in Section 3. I show how projection works in a more familiar case before moving to the hyperboloid case in Section 4. I perform stereographic projection in Euclidean space, then I show how distance works on the sphere and on the projection.

- **Stereographic projection of a hyperboloid** onto the plane in Section 4. I perform stereographic projection, but in Minkowski space, then I show how distance works on the hyperboloid and on the projection.

- **Finding the Poincaré disk distance function** in Section 5. I show that the familiar distance function (1) in the Poincaré disk is not arbitrary. In particular, I conclude that the projected hyperboloid distance function is just the Poincaré disk distance function.

## 2. A Stereographic Perspective

Imagine that you've woken up in the middle of the night and you can't move a muscle no matter how hard you try.

In fact, all you *can* do is stare up at your ceiling in the pitch black dark. Even more alarmingly, you only see three curvy neon-lit lines that come together make a warped triangle on your ceiling (shown on the left in Figure 3). Because you have no other choice, you stare at the triangle and ponder your life.

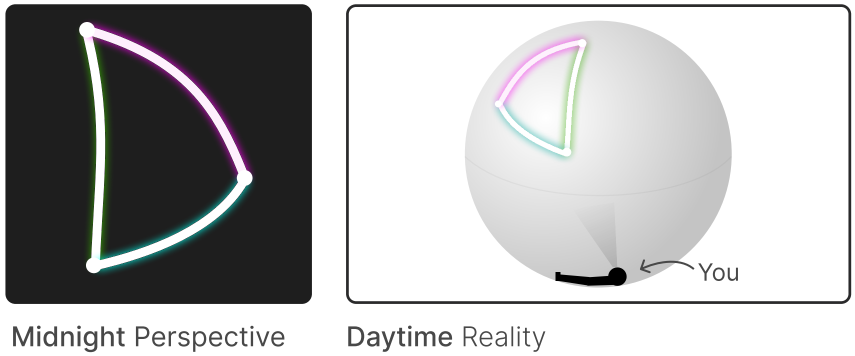


Figure 3: A triangle on a sphere from two perspectives: your first-person point of view at midnight and the three-dimensional reality.

A few hours go by, the sun starts to rise, and glimmers of light shine through your window blinds. Oddly enough, you start to see glares all around you. There seems to be a glass barrier surrounding you. Now that you're depth perception is back, you find the reality of your situation. You are lying down on the bottom of a glass sphere (see Figure 3 on the right). And the sphere has straight neon-colored lines drawn on top of it!

But you swore that in the dark it was just a warped triangle on your flat ceiling! But now it's clear that you were seeing straight lines on a curved surface. This situation describes stereographic projection. In the dark, your bottom perspective (Figure 3) still captures a great deal about the triangle, but is much simpler without the third dimension.

Stereographic projection (your perspective) preserves the angles and distances from the original model [3]. In this scenario, stereographic projection is a conformal representation of the sphere [3]. If distance makes more sense on the sphere, I can think of lines on the sphere, and know that the projection inherits the same properties.

For the purposes of this paper, I will use stereographic projection to show where the Poincaré disk distance function comes from.

## 3. Sphere Stereographic Projection

Before I use stereographic projection on a hyperboloid (Section 4), I'll first show the easier case: stereographic projection of a sphere. Then, I'll show the distance function on the sphere and on the projection.

### *3.1. Projection*

Our sphere defined by $1 = x^2 + y^2 + z^2$ lives in three-dimensional Euclidean space (see Figure 4A) with the Euclidean line element

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{2}$$

A point on our sphere is represented by $(x, y, z)$ with parameters $\theta$ and $\phi$ as

$$D = (\sin(\theta)\sin(\phi), \cos(\theta)\sin(\phi), \cos(\phi)). \tag{3}$$

Like in Section 2, you first take the bottom perspective $B$ looking up to a point on the sphere $D$. I will define $B = (0, 0, -1)$ and $D$ as the point on the sphere (3) as shown in Figure 4B. The projection $P$ is where the segment $BD$ intersects the $xy$ plane in Figure 4B.

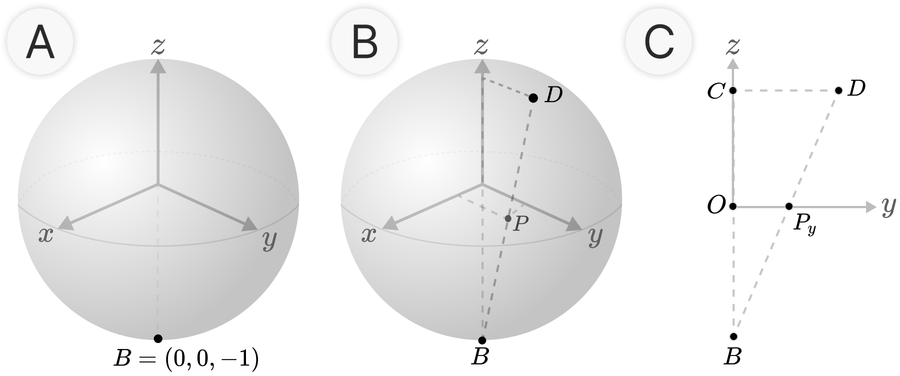


Figure 4: Stereographic projection of a sphere onto the $xy$ plane.

To solve for the projection $P$, I will use similar triangles [4]. If I consider a point $C$ as the $z$ direction of point $D$ and $O$ as the origin, I can construct two triangles: $\triangle DBC$ and $\triangle PBO$. As shown in Figure 4C, the $\triangle PBO$ is nested inside of $\triangle DBC$. The triangles are similar because they have congruent corresponding angles.

Using the similar triangles, I know the corresponding segments are proportional so I can relate $OP$ to $CD$ and $OB$ to $CB$ as $\frac{OP}{CD} = \frac{OB}{CB}$. I know that $OP$ is just the $y$ value of $P$ (notated as $P_y$) and $OB = 1$. I also know that $CD$ is the $y$ value of $D$ as $\cos(\theta)\sin(\phi)$ and $CB = 1 + \cos(\phi)$ to give me

$$\begin{aligned} \frac{OP}{CD} &= \frac{OB}{CB} \\ \frac{P_y}{\cos(\theta)\sin(\phi)} &= \frac{1}{1+\cos(\phi)} \\ P_y &= \frac{\cos(\theta)\sin(\phi)}{1+\cos(\phi)} \\ &= \cos(\theta)\tan\left(\frac{\phi}{2}\right). \end{aligned} \tag{4}$$

Then, I can solve for the $x$ direction of $P$ (notated as $P_x$) with the same method, but where $CD$ is now the $x$ value of $D$ as $\sin(\theta)\sin(\phi)$ to give me

$$\begin{aligned} \frac{OP}{CD} &= \frac{OB}{CB} \\ \frac{P_x}{\sin(\theta)\sin(\phi)} &= \frac{1}{1+\cos(\phi)} \\ P_x &= \frac{\sin(\theta)\sin(\phi)}{1+\cos(\phi)} \\ &= \sin(\theta)\tan\left(\frac{\phi}{2}\right). \end{aligned} \tag{5}$$

Putting the pieces together, I get that the stereographic projection of the sphere onto the $xy$ plane is

$$\begin{aligned} P &= (P_x, P_y) \\ &= \left(\sin(\theta)\tan\left(\frac{\phi}{2}\right), \cos(\theta)\tan\left(\frac{\phi}{2}\right)\right). \end{aligned} \tag{6}$$

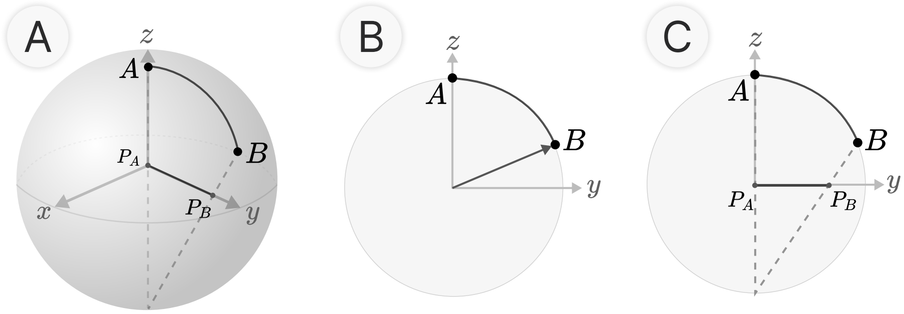


Figure 5: The distance on the sphere is the distance between the same projected points.

### *3.2. Distance*

Now that I have the stereographic projection, I can find the distance function on the sphere and on the projection.

I will consider the distance between two new spherical points $A$ and $B$ as shown in Figure 5A. And to vastly simplify the distance function $d(A, B)$, I will consider the distance between points when $\theta = 0$ which allows me to ignore the $x$ direction as shown in Figure 5C. By ignoring the $x$ direction, I can define points $(y, z)$ only with the $\phi$ parameter like

$$B = (\sin(\phi), \cos(\phi)). \tag{7}$$

If I consider $B$ a vector (Figure 5B), I can take an infinitesimal step away from $B$ on the sphere as the vector $dB$. To see how $\phi$ affects the output, I can take the vector derivative to get

$$\begin{aligned} \frac{dB}{d\phi} &= (\cos(\phi), -\sin(\phi)) \\ dB &= (\cos(\phi), -\sin(\phi))d\phi. \end{aligned} \tag{8}$$

The derivative of $B$ with respect to $\phi$ represents the tangent vector to $B$. So $d\phi$ scaled by the magnitude of the tangent is just $dB$. Since we are in Euclidean space, I can use Euclidean distance to get the magnitude of a vector $v$ as $||v|| = \sqrt{v_x^2 + v_y^2 + v_z^2}$.

I can take the magnitude of the vector in (8) to get that

$$\begin{aligned} dB &= ||(\cos(\phi), -\sin(\phi))||\, d\phi \\ &= \sqrt{\cos^2(\phi) + \sin^2(\phi)}\, d\phi \\ &= d\phi \end{aligned} \tag{9}$$

by using that $y^2 + z^2 = 1$ when $\theta = 0$ and $\sqrt{\cos^2(\phi) + \sin^2(\phi)} = 1$.

From (9), I conclude that distance and $\phi$ are the same. Furthermore, (9) shows that $\phi$ is an angle so I can use trigonometric properties in the distance formulation.

Next, I can integrate (9) between two angles $\phi_A$ and $\phi_B$ (from $A$ and $B$) to get the distance function as

$$\int_{\phi_A}^{\phi_B} d\phi = \phi\Big|_{\phi_A}^{\phi_B}$$

$$d(\phi_A, \phi_B) = \phi_B - \phi_A \tag{10}$$

$$d(A, B) = \sin^{-1}(B) - \sin^{-1}(A) \tag{11}$$

where (10) is the distance given the angles and (11) is the angle conversion to $y$ values of $A$ and $B$.

Then, to get the distance function on the projection $P$, I can transform the coordinates so that the spherical distance function takes projection $P$ as input.

Recall that $P_y = \cos(\theta)\tan\left(\frac{\phi}{2}\right)$ from (4), and due to my simplification of $\theta = 0$, the $P_y$ is just $\tan\left(\frac{\phi}{2}\right)$. I can solve for $\phi$ as $\phi = 2\tan^{-1}(P_y)$ and substitute the angle $\phi$ back into (10) to get

$$d_P(P_A, P_B) = 2\tan^{-1}(P_B) - 2\tan^{-1}(P_A) \tag{12}$$

given $P_A$ and $P_B$ are the projected points as shown in Figure 5C. Finally, we have the distance function $d_P$ for the points on the spherical projection!

## 4. Hyperboloid Stereographic Projection

To get the Poincaré disk distance function, I can apply the logic from the last Section 3, but in Minkowski space where the line element is

$$ds^2 = dx^2 + dy^2 - dz^2 \tag{13}$$

instead of the Euclidean line element (2).

### 4.1. Projection

First, I'll define a hyperboloid as $1 = z^2 - y^2 - x^2$ as shown in Figure 6A. Similar to the Euclidean sphere, the hyperboloid has constant distance from the center to any point on the hyperboloid in Minkowski space.

A point on the hyperboloid $(x, y, z)$ can be defined parametrically by $\theta$ and $\phi$ as

$$D = (\sin(\theta)\sinh(\phi), \cos(\theta)\sinh(\phi), \cosh(\phi))\,. \tag{14}$$

To project the hyperboloid point $D$ onto the $xy$ plane, I can place a bottom perspective $B = (0, 0, -1)$ and define $P$ as the intersection of $BD$ with the $xy$ plane as shown in Figure 6B.

I will apply the same similar triangles reasoning [4] in Section 3.1 to create the triangles $\triangle DBC$ and $\triangle PBO$ where $C$ is the $z$ direction of $D$ and $O$ is the origin as shown in Figure 6C.

Then, I know the corresponding segments are proportional so I can relate $OP$ to $CD$ and $OB$ to $CB$ as $\frac{OP}{CD} = \frac{OB}{CB}$ as shown in Figure 6C. I know that $OP$ is just the $y$ value of $P$

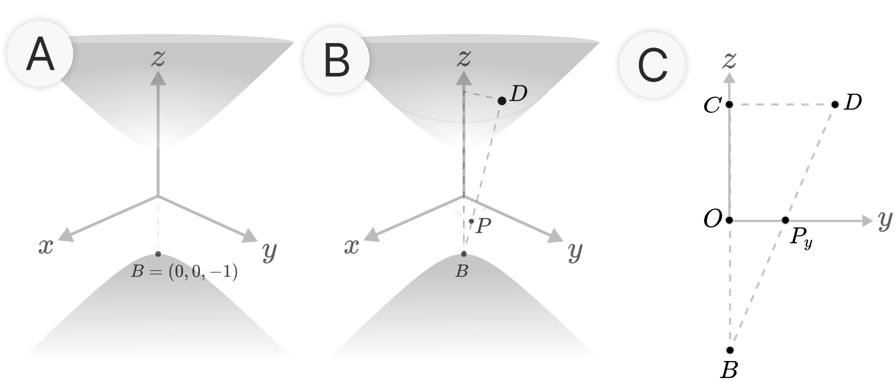


Figure 6: Stereographic projection of a hyperboloid onto the $xy$ plane.

(notated as $P_y$) and $OB = 1$. I also know that $CD$ is the $y$ value of $D$ as $\cos(\theta)\sinh(\phi)$ and $CB = 1 + \cosh(\phi)$ to give me

$$\begin{aligned}
\frac{OP}{CD} &= \frac{OB}{CB} \\
\frac{P_y}{\cos(\theta)\sinh(\phi)} &= \frac{1}{1+\cosh(\phi)} \\
P_y &= \frac{\cos(\theta)\sinh(\phi)}{1+\cosh(\phi)} \\
&= \cos(\theta)\tanh\left(\frac{\phi}{2}\right).
\end{aligned} \tag{15}$$

Then, I can solve for the $x$ direction of $P$ (notated as $P_x$) with the same method, but where $CD$ is now the $x$ value of $D$ as $\sin(\theta)\sinh(\phi)$ to give me

$$\begin{aligned}
\frac{OP}{CD} &= \frac{OB}{CB} \\
\frac{P_x}{\sin(\theta)\sinh(\phi)} &= \frac{1}{1+\cosh(\phi)} \\
P_x &= \frac{\sin(\theta)\sinh(\phi)}{1+\cosh(\phi)} \\
&= \sin(\theta)\tanh\left(\frac{\phi}{2}\right).
\end{aligned} \tag{16}$$

Putting the pieces together, I get that the stereographic projection, the Poincaré disk, is

$$\begin{aligned}
P &= (P_x, P_y) \\
&= \left(\sin(\theta)\tanh\left(\frac{\phi}{2}\right), \cos(\theta)\tanh\left(\frac{\phi}{2}\right)\right).
\end{aligned} \tag{17}$$

### *4.2. Distance*

Now that I have the stereographic projection from hyperboloid to Poincaré disk, I can find the distance function on the hyperboloid and on the projection.

Just like in Section 3.2, I will consider the distance between two new points $A$ and $B$ as shown in Figure 7A. To vastly simplify the distance function $d(A, B)$, I will consider the distance between

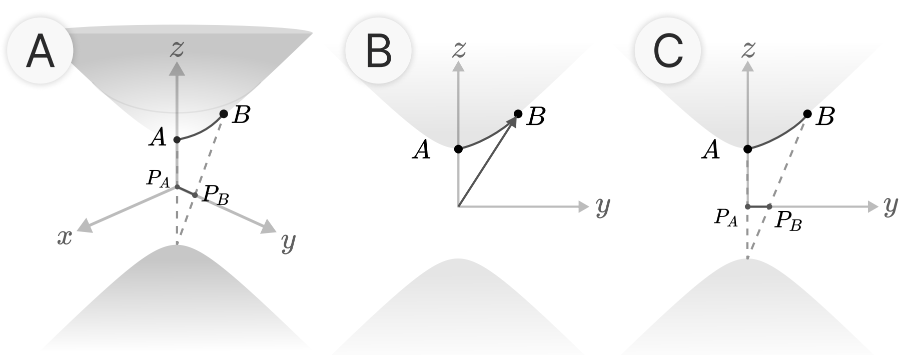


Figure 7: The distance on the hyperboloid is the same as on the Poincaré disk projection.

points when $\theta = 0$ which allows me to ignore the $x$ direction as shown in Figure 7C. The point can be represented as $(0, \sinh(\phi), \cosh(\phi))$ when $\theta = 0$ so I can represent the two-dimensional $(y, z)$ vector as

$$B = (\sinh(\phi), \cosh(\phi)). \tag{18}$$

as shown in Figure 7B.

From the vector $B$ in (18), I can take an infinitesimal step from $B$ as $dB$. Relating $dB$ and $d\phi$ I get the following vector derivative

$$\begin{aligned} \frac{dB}{d\phi} &= (\cosh(\phi), \sinh(\phi)) \\ dB &= (\cosh(\phi), \sinh(\phi))d\phi. \end{aligned} \tag{19}$$

The derivative of $B$ with respect to $\phi$ represents the tangent vector to $B$. So the magnitude of the tangent vector scales $d\phi$ to become $dB$. Since we are in Minkowski space, the magnitude of a vector $v$ is $||v|| = \sqrt{v_x^2 + v_y^2 - v_z^2}$.

I can take the magnitude of the vector in (19) to get that

$$\begin{aligned} dB &= ||(\cosh(\phi), \sinh(\phi))||\, d\phi \\ &= \sqrt{\cosh^2(\phi) - \sinh^2(\phi)}\, d\phi \\ &= d\phi \end{aligned} \tag{20}$$

since $y^2 - z^2 = 1$ when $\theta = 0$ and $\sqrt{\cosh^2(\phi) - \sinh^2(\phi)} = 1$.

From (20), I conclude that distance and $\phi$ are the same. Furthermore, (20) shows that $\phi$ is an angle so I can use hyperbolic trigonometry in the distance formulation.

Then, I can integrate (20) from $\phi_A$ to $\phi_B$ to get the hyperboloid distance function as

$$\int_{\phi_A}^{\phi_B} d\phi = \phi\Big|_{\phi_A}^{\phi_B}$$

$$d(\phi_A, \phi_B) = \phi_B - \phi_A \tag{21}$$

$$d(A, B) = \sinh^{-1}(B) - \sinh^{-1}(A) \tag{22}$$

where (21) is the distance given the angles and (22) is the angle conversion to $y$ values of $A$ and $B$.

Finally, I can convert the units so that I can input projection points $P$ into the hyperboloid distance function (21). Since I only consider when $\theta = 0$, I am only concerned with the $y$ value of $P$ as $P_y = \tanh^{-1}\left(\frac{\phi}{2}\right)$. I can solve for $\phi$ as $\phi = 2\tanh^{-1}(P_y)$ and substitute the angle $\phi$ back into (21) like

$$d_P(P_A, P_B) = 2\tanh^{-1}(P_B) - 2\tanh^{-1}(P_A) \tag{23}$$

where $P_A$ and $P_B$ are the $y$ values of the projected $A$ and $B$ points.

Then, I can use the fact that $\tanh^{-1}(z) = \frac{1}{2}\left[\ln(1+z) - \ln(1-z)\right]$ from [5] to simplify (23)

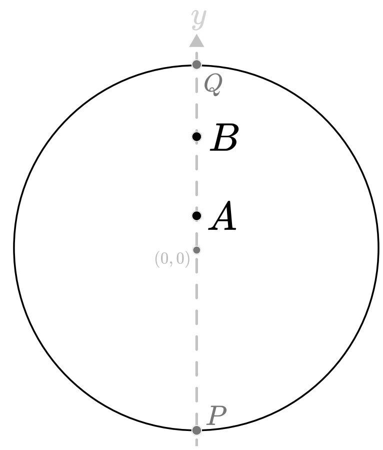


Figure 8: Points on the Poincaré disk $y$ axis.

to

$$\begin{aligned}
d_P(P_A, P_B) &= 2\tanh^{-1}(P_B) - 2\tanh^{-1}(P_A) \\
&= 2\left(\frac{1}{2}\left[\ln(1+P_B) - \ln(1-P_B)\right]\right) - 2\left(\frac{1}{2}\left[\ln(1+P_A) - \ln(1-P_A)\right]\right) \\
&= \ln\left(\frac{1+P_B}{1-P_B}\right) - \ln\left(\frac{1+P_A}{1-P_A}\right) \\
&= \ln\left(\frac{1+P_B}{1-P_B} \times \frac{1-P_A}{1+P_A}\right) \\
&= \ln\left(\frac{1+P_B}{1+P_A} \times \frac{1-P_A}{1-P_B}\right). \qquad (24)
\end{aligned}$$

## 5. Finding the Poincaré disk distance function

Finally, I will show that the original Poincaré disk distance function (1) is actually the same as what we got in (24) for the projection.

Since I restricted $\theta = 0$ and kept $\phi$ positive in my formulation, I only consider positive $y$ values in the projection. Starting from the original Poincaré disk, I can define points $A$ and $B$ on the vertical $y$ axis and boundary points $P$ and $Q$ as shown in Figure 8.

Then, I can find the Eucldean measures defined in the original Poincaré disk distance function (1). Since the Euclidean distance from a boundary point to the center is one, it's straightforward to see in Figure 8 that $AQ = 1 - A_y$, $BQ = 1 - B_y$, $BP = 1 + B_y$, and $AP = 1 + A_y$.

Substituting the Euclidean segment measures back into (1) I get that

$$\begin{aligned}
d(A, B) &= \left|\ln\left(\frac{AQ}{AP} \times \frac{BP}{BQ}\right)\right| \\
&= \ln\left(\frac{1-A_y}{1-B_y} \times \frac{1+B_y}{1+A_y}\right). \qquad (25)
\end{aligned}$$

By using my previous notation (where $A_y$ is just the projection $P_A$ and $B_y$ is just the projection $P_B$) and reordering the multiplication, I finally get

$$\ln\left(\frac{1+P_B}{1+P_A} \times \frac{1-P_A}{1-P_B}\right) \qquad (26)$$

which is the exact same as the projected distance function we found in (24)!

The Poincaré disk distance function is just the distance function after stereographically projecting the hyperboloid onto the $xy$ plane in Minkowski space!

## 6. Conclusion

By showing distance on the hyperboloid and on the stereographic projection, we've found where the Poincaré disk distance function comes from.

The distance function is still not very simple, but at least you can see that the function is not arbitrary! The Poincaré disk distance function comes from a very logical starting point in Minkowski space.

In the future, generalizing the arguments in this paper would make a more comprehensive, but much lengthier, explanation. In particular, I only considered points such that $\theta = 0$ to vastly simplify the distance function for presentation.

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## References

1. T. Dray, "Models of Non-Euclidean Geometry: The Poincaré Disk," https://books.physics.oregonstate.edu/MNEG/pintro.html.
2. E. C. Wallace and S. F. West, *Roads to geometry* (Waveland Press, 2015).
3. C. R. Graham *et al.*, "Conformal powers of the Laplacian via stereographic projection," SIGMA. Symmetry, Integrability Geom. Methods Appl. **3**, 121 (2007).
4. T. Dray, "Models of Non-Euclidean Geometry: Stereographic Projection," https://books.physics.oregonstate.edu/MNEG/sterproj.html.
5. E. W. Weisstein, "Inverse Hyperbolic Tangent," Wolfram MathWorld (2002). https://mathworld.wolfram.com/InverseHyperbolicTangent.html.